# Unbalanced Random Matching Markets

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# Outline

# The Setting

1. Let $\mathcal{S} = \{1, 2, \ldots, n\}$ be the set of students planning to apply for higher studies after school, and $\mathcal{U} = \{1, 2, \ldots, n\}$ be the set of universities (here each university takes only one student).

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4. A matching $\mu$ is *unstable* if there is a student $s$ and a university $u$ s.t. $u \succ_s \mu(s)$ & $s \succ_u \mu(u)$. Otherwise it is stable. The set of stable matchings is called the core.

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# The Gale-Shapley Theorem

## The Theorem

Consider a group of $n$ students and $n$ universities, and suppose that every student [and every university] has a complete preference list of all the universities [students] in the group. Then, there is a stable matching between the students and the university.

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## The Deferred Algorithm

1. Every unmatched student applies to his/her most preferred university which has not already rejected him/her. If no new application is made, output the current matching.
2. If a university has multiple applications, it tentatively keeps its most preferred student and rejects the rest. Go to step 1.

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# Average Ranks

Let $\text{Rank}_s(u)$ denote the rank of $u$ in the preference list $\succ_s$ of $s$ and let $\text{Rank}_u(s)$ have a symmetrical definition.
Given a matching $\mu$, the average students rank of their universities is given by:

$$R_{STUDENTS}(\mu) = \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} \text{Rank}_s(\mu(s))$$

Similarly, average universities rank of their students is given by:

$$R_{UNIVERSITIES}(\mu) = \frac{1}{|\mathcal{U}|} \sum_{u \in \mathcal{U}} \text{Rank}_u(\mu(u))$$

# Pittel (1989), Knuth-Motwani-Pittel (1990)

In a random matching market with $n$ students and $n$ universities, the fraction of agents that have multiple stable matchings converges to 1 as $n \to \infty$. Furthermore,

$$R_{STUDENTS}(SOSM) \xrightarrow{p} log\ \ n$$

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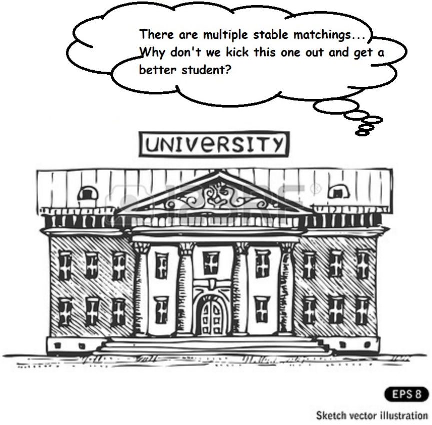
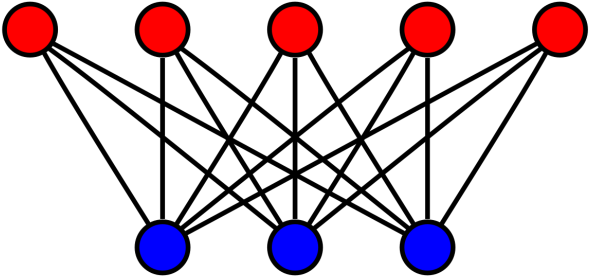


# Outline

# Questions we are looking at...

Suppose that now there are $n$ students and $n + k$ universities. If the preferences are random and students again are applying, what is the likely number of stable matchings and also what is the likely average students' rank of their universities?



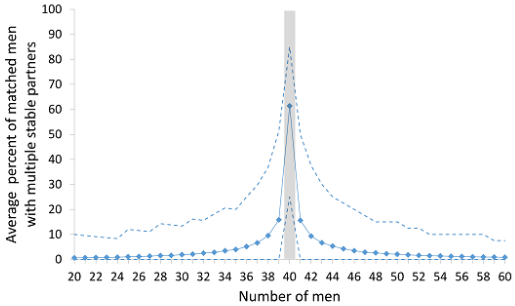



Figure 1: Percent of men with multiple stable partners, in random markets with 40 women and a varying number of men. The main line indicates the average over 10,000 realizations. The dotted lines indicate the top and bottom 2.5th percentiles.

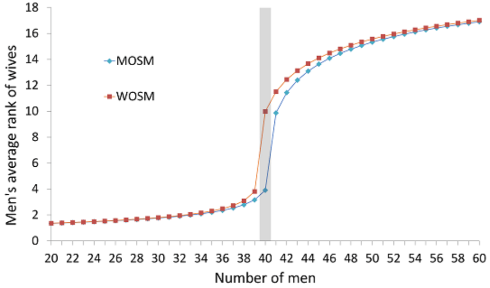



Figure 2: Men's average rank of wives under MOSM and WOSM in random markets with 40 women and varying number of men. The lines indicate the average over 10,000 realizations.

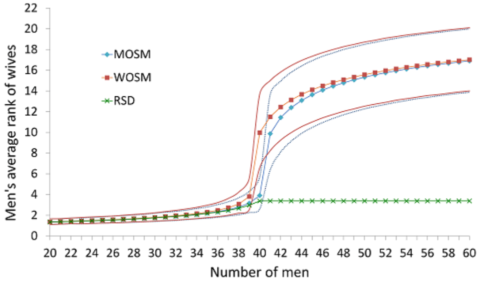



Figure 3: Men's average rank of wives under MOSM and WOSM in random markets with 40 women and a varying number of men. The main lines indicate the average over matched mens' average rank of wives in all 10,000 realizations. The dotted lines indicate the top and bottom 2.5th percentile of the 10,000 realizations. The labeled RSD gives the men's average rank under the Random Serial Dictatorship mechanism.

# Unequal numbers lead to a small core

Fix $\epsilon > 0$. Consider a sequence of random matching markets, indexed by $n$, with $n$ students and $n+k$ universities, for arbitrary $k = k(n) > 1$. With high probability, we have that

1. the fraction of students and fraction of universities who have multiple stable partners tend to zero as $n \to \infty$, and
2. the students' average rank of universities is almost the same under all stable matchings:

$$(1-\epsilon) R_{STUDENT}(SOSM) \leq R_{STUDENT}(UOSM)$$

$$R_{STUDENT}(UOSM) \leq (1+\epsilon) R_{STUDENT}(SOSM)$$

# Agents on the shorter side have an advantage

Consider a sequence of random matching markets, indexed by $n$, with $n$ students and $n+k$ universities, for arbitrary $k = k(n) \geq 1$. With high probability, the following holds for every stable matching $\mu$:

$$R_{STUDENTS}(\mu) \leq (1+\epsilon)\frac{n+k}{n} log \frac{n+k}{k}$$

$$R_{UNIVERSITIES}(\mu) \geq \frac{n}{[1+(1+\epsilon)\frac{n+k}{n} log \frac{n+k}{k}]}$$

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# Adding a few extra universities

1. Every student has the option of choosing one of the new universities.
2. Only some students will prefer the new universities.
3. However, changing the allocation of some students will lead to changing allocation of many students because if some students are made better off, some universities are made worse off creating more options for the students.
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# Proof Sketch

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## Going from SOSM to UOSM

Until the rejection chain of every university leads to a terminal phase, at every step, for a university $u$ (with matched partner $s$), trigger an artificial rejection chain which can end in:

- Improvement Phase: $u$ accepts a student it prefers over $s$ (Finding a new stable matching closer to UOSM)
- Terminal Phase: An unmatched university is applied to (In this case, all the universities in the chain are already matched its most preferred stable student)

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1. The number of proposals is stochastically dominated by the number of draws in the coupon collector's problem, when $n$ distinct coupons must be drawn from $n + k$ coupons.

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2. This latter quantity is a sum of geometric $\frac{(n+k-i+1)}{(n+k)}$ random variables for $i \in \{1, 2, \ldots, n\}$, whose mean is

$$(n+k)log\left(\frac{n+k}{k}\right)\left(1+O\left(\frac{1}{klog\left(\frac{n+k}{k}\right)}\right)\right)$$

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3. $\frac{1}{klog\left(\frac{n+k}{k}\right)}$ is a monotone decreasing function and hence the error term $O\left(\frac{1}{klog\left(\frac{n+k}{k}\right)}\right) = O\left(\frac{1}{log\ \ n}\right)$ which vanishes in the limit.

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3. It can be shown that unequal numbers lead to a small core in this setting as well.
4. Agents on the shorter side still have an advantage in this setting. However, the results deteriorate slightly depending on the values of $q_u$.

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4. Some of the results can be extended to many-to-one markets.

# Thank you