Faithful Homomorphisms and PIT – a short survey

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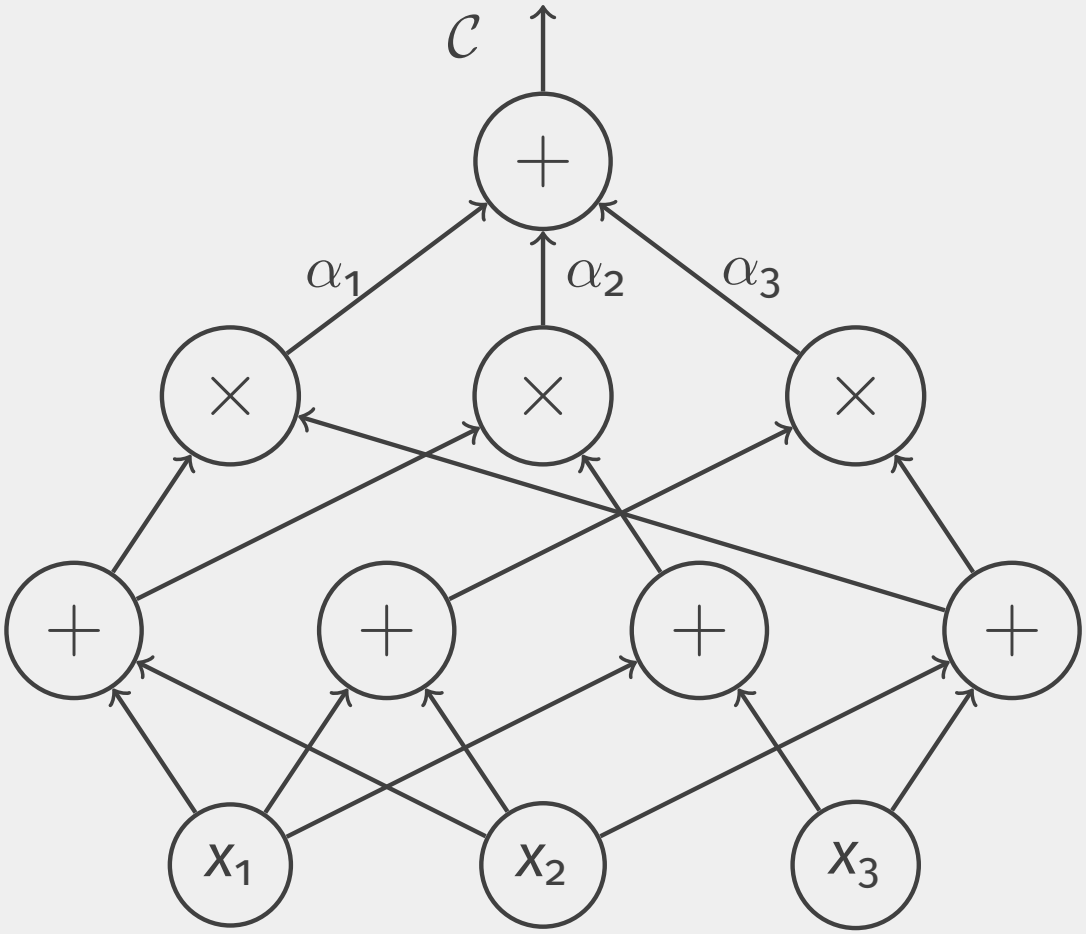
based on joint work with *Ramprasad Saptharishi*

IIT Kanpur

November 3, 2018

Introduction

Polynomial Identity Testing



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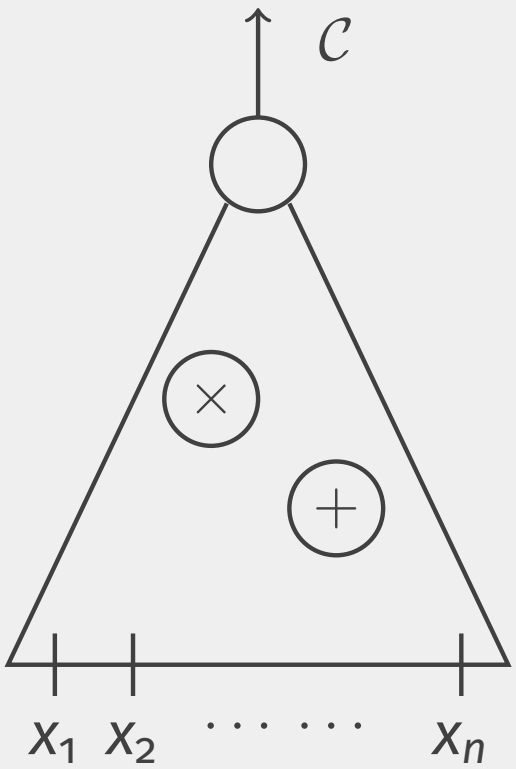
What we need: Reduce no. of variables (preferably constant)
Keep degree under control (poly-bounded)
Preserve non-zeroness

A Special Setting: Can we do better?



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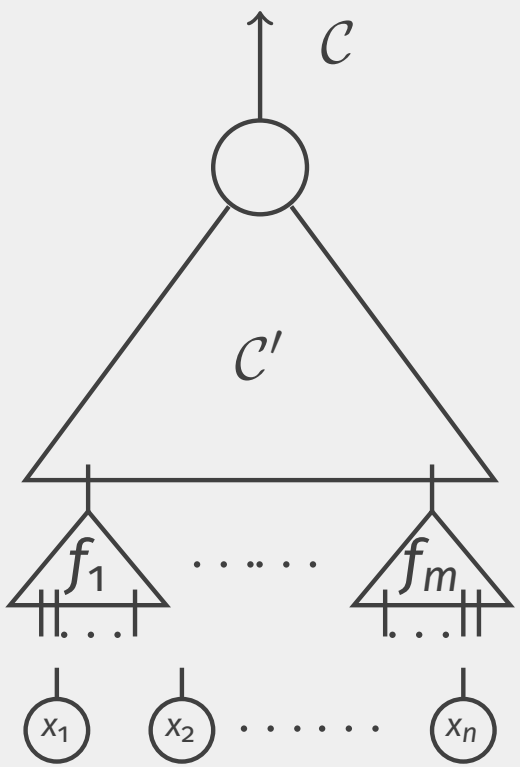

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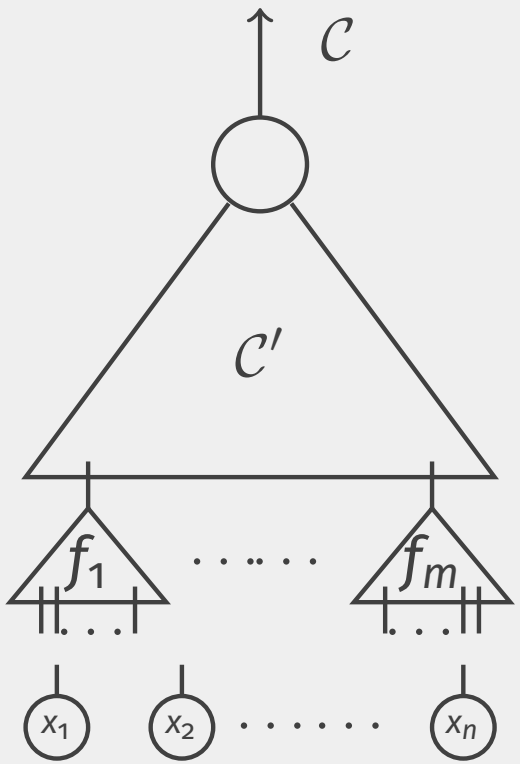

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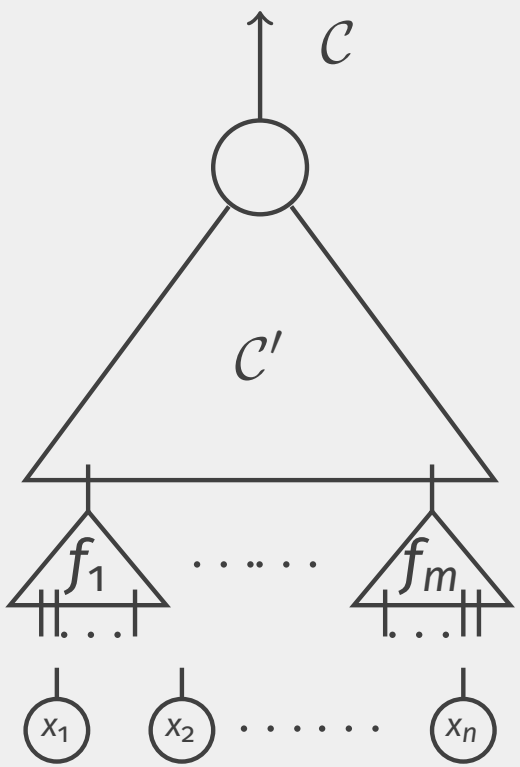

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Can we do any better than trivial?

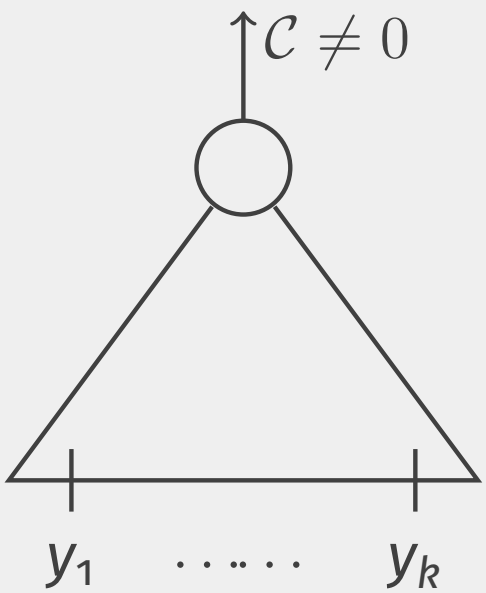
Preliminaries

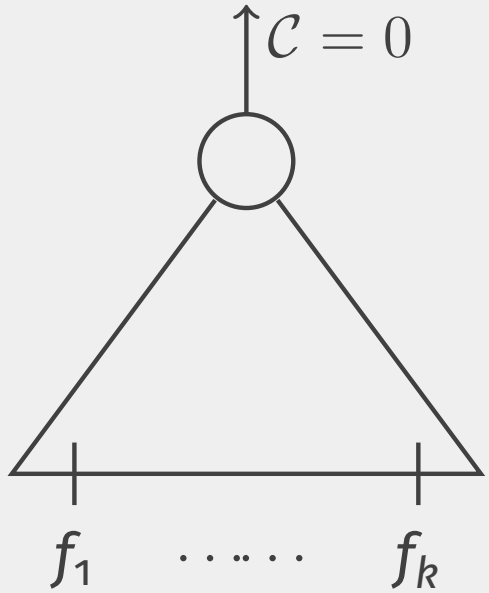
Algebraic Independence

Definition: Suppose $\{f_1, \ldots, f_k\} \subseteq \mathbb{F}[x_1, x_2, \ldots, x_n]$. They are said to be algebraically dependent if there exists $A \in \mathbb{F}[y_1, \ldots, y_k]$ such that

$$A(y_1, \ldots, y_k) \neq 0; \qquad A(f_1, \ldots, f_k) = 0.$$

Otherwise, they are said to be algebraically independent.





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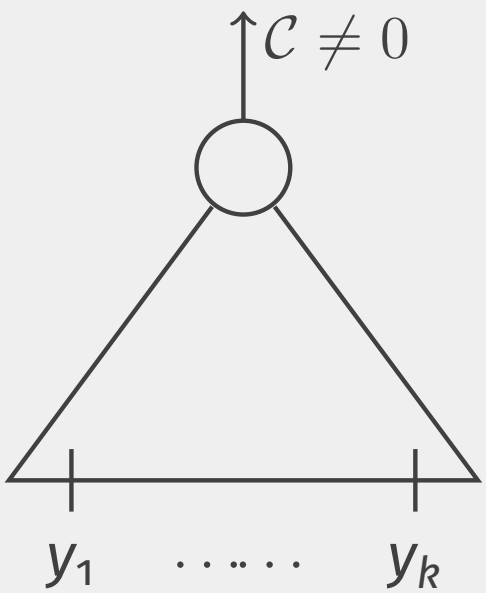
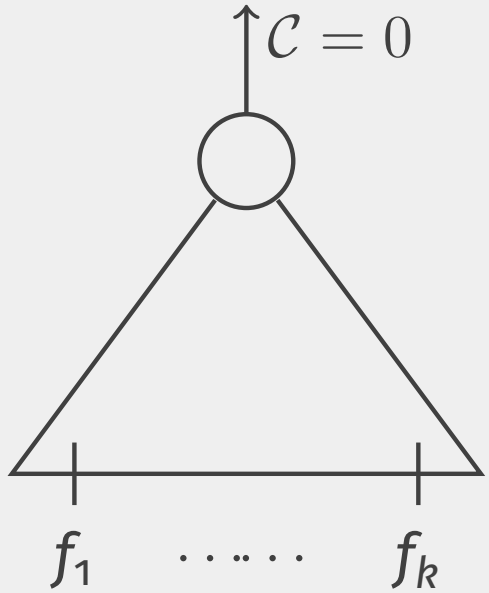


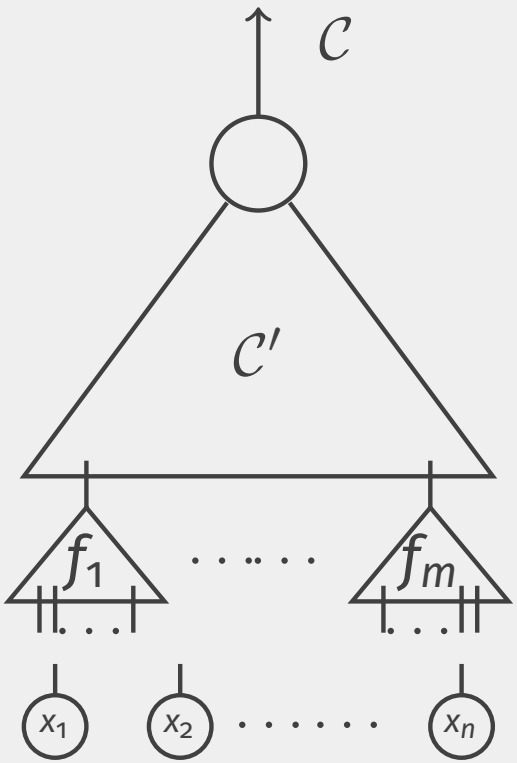


Algebraic Rank of $\{f_1, \ldots, f_m\}$

Size of the maximal algebraically independent subset of $\{f_1, \ldots, f_m\}$.

Our Setting: How is it different?

$$\mathcal{C} = \mathcal{C}'(f_1, \ldots f_m) : \text{algrank}(f_1, \ldots, f_m) = k << n$$



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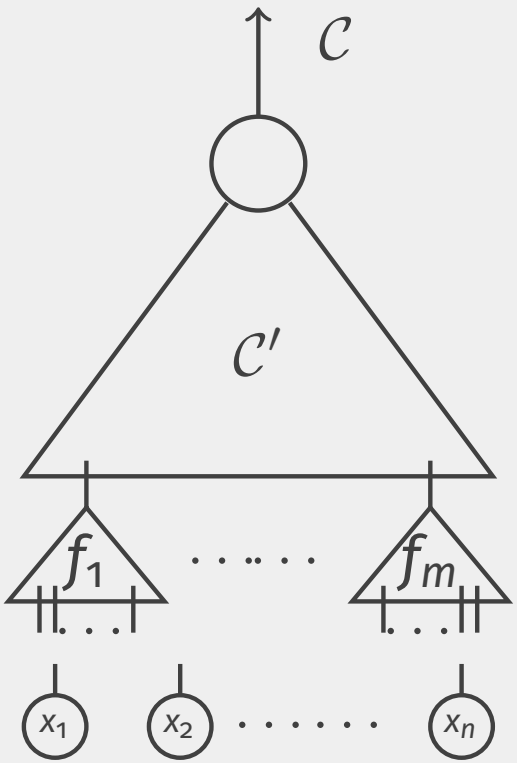

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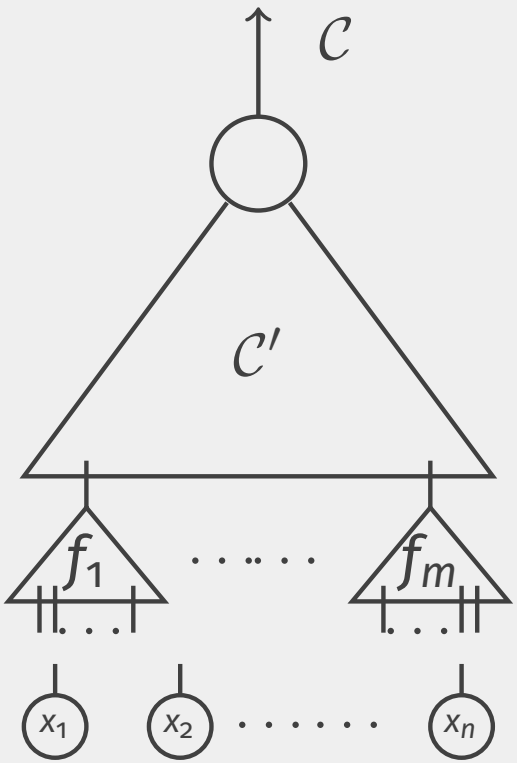

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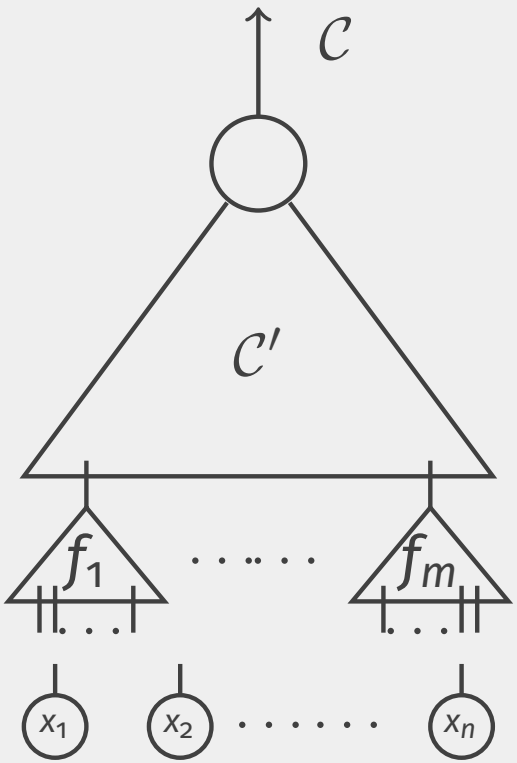

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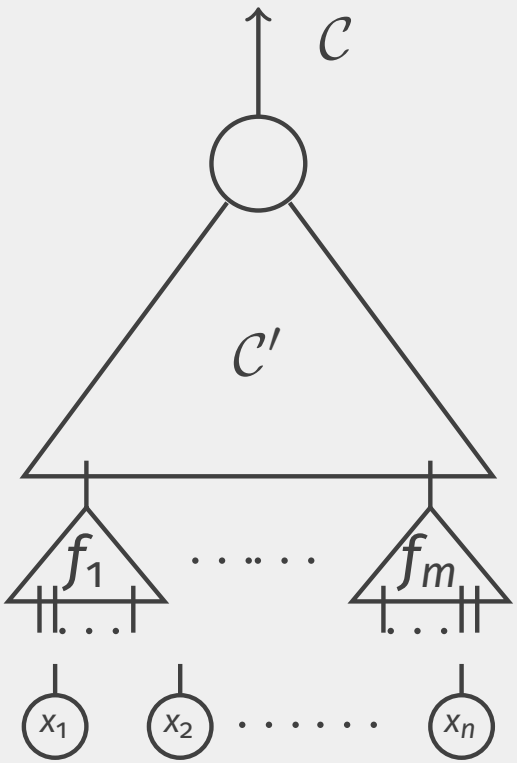

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What is allowed: Substitute x_is

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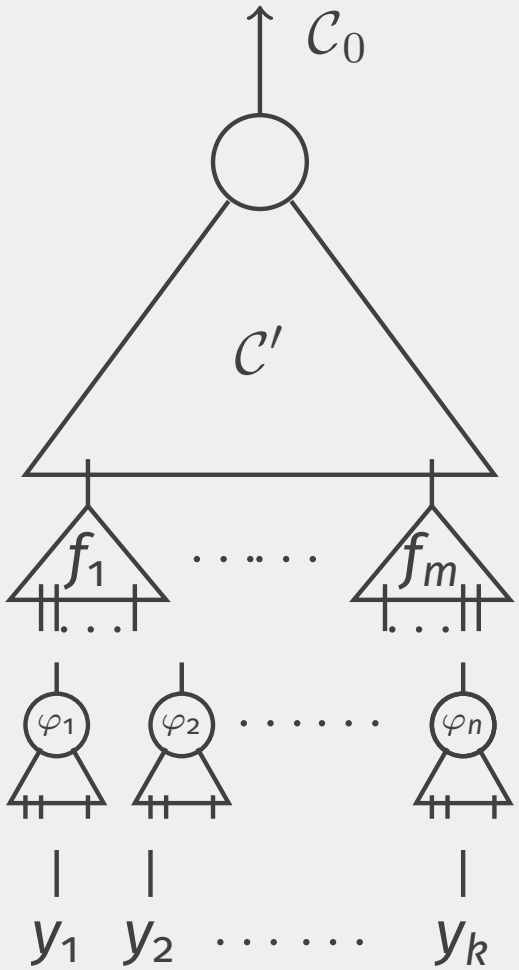

$x_i \to \varphi_i(y_1, \ldots, y_k)$

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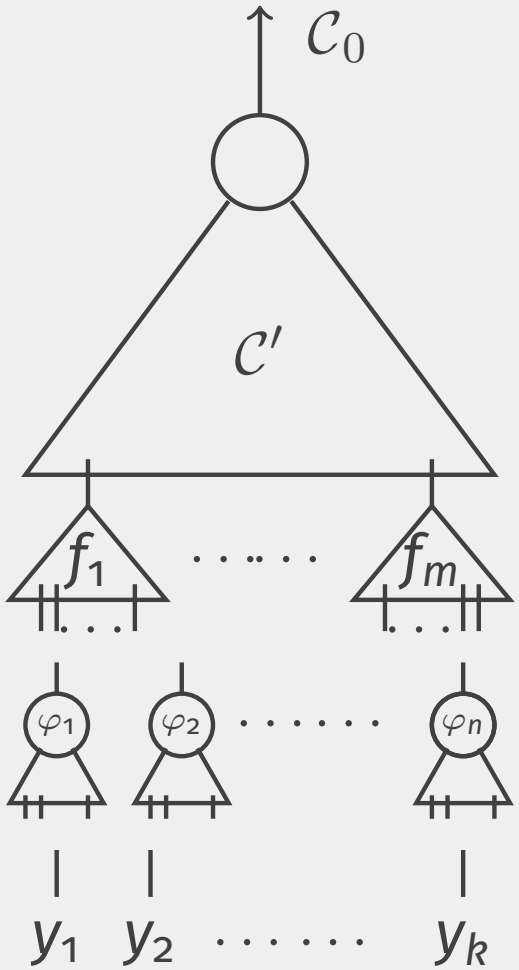

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For $k = m$, **sufficient property** of φ_i's:

$$g_i = f_i(\varphi_1, \ldots, \varphi_n)$$

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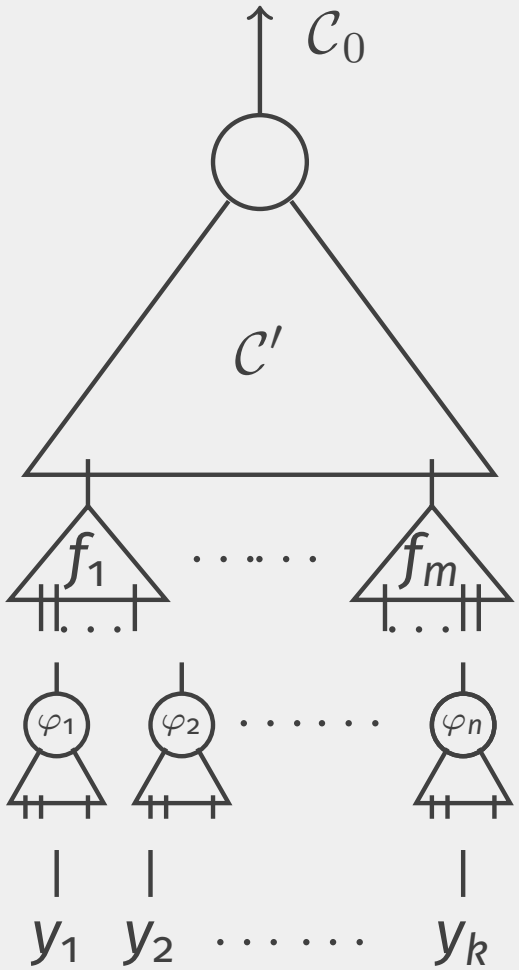

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Amazing fact [BMS'11, ASSS'12]:
Works even in the general case ($k < m$).

Faithful Maps

Definition: Let $\{f_1, \ldots, f_m\} \subseteq \mathbb{F}[\mathbf{x}]$ with $\operatorname{algrank}(f_1, \ldots, f_m) = k$. A map

$$\varphi : \{x_1, \ldots x_n\} \to \mathbb{F}[y_1, y_2, \ldots, y_k]$$

is said to be faithful if $\operatorname{algrank}(g_1, \ldots, g_m) = k$.
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Why should one believe that such maps can exist?
What we know: If $\{f_1, \ldots, f_k\}$ are algebraically independent, then

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What we are asking for: A better way of seeing this isomorphism.
That is, are there $g_1, \ldots, g_k \in \mathbb{F}[y_1, \ldots y_k]$ such that
$g_i = f_i(\varphi(x_1), \ldots, \varphi(x_n))$ and

$$\mathbb{F}(f_1, \ldots, f_k) \cong \mathbb{F}(g_1, \ldots, g_k)?$$

Constructing Faithful Maps are enough

Lemma [BMS'11, ASSS'12]

If φ is a faithful map for $\{f_1, \ldots, f_m\} \subseteq \mathbb{F}[\mathbf{x}]$, then for any $\mathcal{C}$

$$\mathcal{C}(f_1, \ldots, f_m) \neq 0 \implies \mathcal{C}(f_1 \circ \varphi, \ldots, f_m \circ \varphi) \neq 0$$

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Proof: Let $g_i = f_i(\varphi(x_1), \ldots, \varphi(x_n))$.

$$\begin{aligned}
&\mathcal{C}[f_1, \ldots, f_m] \neq 0 \Rightarrow \mathcal{C}(f_1, \ldots, f_m) \neq 0 \\
&\Rightarrow \mathcal{C}(f_1, \ldots, f_m) \cdot R(f_1, \ldots, f_m) = 1 \\
&\Rightarrow \mathcal{C}(f_1, \ldots, f_m) \cdot R'[f_1, \ldots, f_m] = Q[f_1, \ldots, f_k] \\
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&\qquad\qquad = Q[g_1, \ldots, g_k] \neq 0 \\
&\Rightarrow \mathcal{C}(g_1, \ldots, g_m) = \mathcal{C}(f_1 \circ \varphi, \ldots, f_m \circ \varphi) \neq 0
\end{aligned}$$

Useful fact

$\mathbb{F}(f_1, \ldots, f_m)$ | $\mathbb{F}(f_1, \ldots, f_k)$ is an algebraic extension. Thus, $R(f_1, \ldots, f_m)$

$$= \frac{R'[f_1, \ldots, f_m]}{Q[f_1, \ldots, f_k]}$$

Linear Rank Extractors [GR'05]

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix} \times \begin{bmatrix} s & \dots & s^k \\ (s^2)^1 & \dots & (s^2)^k \\ \vdots & & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ (s^n)^1 & \dots & (s^n)^k \end{bmatrix} = \begin{bmatrix} \\ A' \\ \\ \end{bmatrix}$$

A has full $\mathbb{F}$-rank

$\implies$ $\det(A')$ is a non-zero polynomial in s of degree at most nk^2

$\implies$ A' has full $\mathbb{F}(s)$-rank

$\implies$ For some $s \in \{0, \dots, nk^2\}$ A' has full $\mathbb{F}$-rank

Capturing Algebraic Rank via Linear Rank

For $f_1, f_2, \ldots, f_m \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ and $\mathbf{f} = (f_1, f_2, \ldots, f_m)$,

$$\mathbf{J}_{\mathbf{x}}(\mathbf{f}) = \begin{bmatrix} \partial_{x_1}(f_1) & \partial_{x_2}(f_1) & \cdots & \partial_{x_n}(f_1) \\ \partial_{x_1}(f_2) & \partial_{x_2}(f_2) & \cdots & \partial_{x_n}(f_2) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1}(f_m) & \partial_{x_2}(f_m) & \cdots & \partial_{x_n}(f_m) \end{bmatrix}$$

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The Jacobian Criterion

If $\mathbb{F}$ has characteristic zero, the algebraic rank of $\{f_1, f_2, \ldots, f_m\}$ is equal to the linear rank of its Jacobian matrix.

Characteristic Zero Fields: [Jac'41], [BMS'11], [ASSS'12]

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What we need: $\mathbf{J_x}(\mathbf{f})$ has full rank $\implies$ $\mathbf{J_y}(\mathbf{f} \circ \varphi)$ has full rank

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Then for $M_\varphi[i, j] = a_{ij}$,

$$\begin{bmatrix} \\ \mathbf{J_y}(\mathbf{f} \circ \varphi) \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \varphi(\mathbf{J_x}(\mathbf{f})) \\ \\ \end{bmatrix} \times \begin{bmatrix} \\ M_\varphi \\ \\ \end{bmatrix}$$

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Guarantee: If $\mathcal{C} = \mathcal{C}'(f_1,\ldots,f_m) \neq 0$, then for some $\varphi \in \mathcal{F}$,
$\mathcal{C}_0 = \mathcal{C}'(f_1 \circ \varphi,\ldots,f_m \circ \varphi) \neq 0.$

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When is it useful? k is a constant and $|\mathcal{S}| \leq \text{poly}(n)$.

Some Instantiations

The PIT Algorithm

Input: $\mathcal{C} = \mathcal{C}'(f_1, \ldots, f_m)$, $k \geq \text{algrank}(f_1, \ldots, f_m)$

Step 1: For every $\varphi \in \mathcal{F}$ not tried yet.
Run trivial PIT on $\mathcal{C}_0 = \mathcal{C}'(f_1 \circ \varphi, \ldots, f_m \circ \varphi)$
If $\mathcal{C}_0 \neq 0$, return "NOT ZERO".

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Some Examples: $\mathcal{C} = \mathcal{C}'(f_1, \ldots, f_m)$ where

- each f_i is a sparse polynomial
- each f_i is a product of multilinear, variable disjoint, sparse polynomials
- each f_i is a product of linear polynomials

Arbitrary Fields: The PSS Criterion [PSS'16]

What goes wrong over arbitrary fields?

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$$\mathbf{J}_{x,y} = \begin{bmatrix} y^{p-1} & (p-1)xy^{p-2} \\ (p-1)x^{p-2}y & x^{p-1} \end{bmatrix}$$

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Finite Characteristic: Entries in "inverse" have denominators that are partial derivatives of some annihilators - Can become zero.

Looking Further in the Taylor Expansion

For any $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ and $\mathbf{z} \in \mathbb{F}^n$,

$$f(\mathbf{x}+\mathbf{z}) - f(\mathbf{z}) = \underbrace{x_1 \cdot \partial_{x_1} f + \cdots + x_n \cdot \partial_{x_n} f}_{\text{Jacobian}} + \text{higher order terms}$$

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A New Operator

For any $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$,

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$$\hat{\mathcal{H}}(\mathbf{f}) = \begin{bmatrix} \ldots & \mathcal{H}_t(f_1) & \ldots \\ \ldots & \mathcal{H}_t(f_2) & \ldots \\ & \vdots & \\ \ldots & \mathcal{H}_t(f_k) & \ldots \end{bmatrix}.$$

The PSS Criterion

A given set of polynomials $\{f_1, f_2, \ldots, f_k\} \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ is algebraically independent if and only if for a random $\mathbf{z} \in \mathbb{F}^n$, $\{\mathcal{H}_t(f_1), \mathcal{H}_t(f_2), \ldots, \mathcal{H}_t(f_k)\}$ are linearly independent in

$$\frac{\mathbb{F}(\mathbf{z})[x_1, x_2, \ldots, x_n]}{\mathcal{I}_t}$$

where t is the inseparable degree of $\{f_1, f_2, \ldots, f_k\}$ and

$$\mathcal{I}_t = \langle \mathcal{H}_t(f_1), \mathcal{H}_t(f_2), \ldots, \mathcal{H}_t(f_k) \rangle_{\mathbb{F}(\mathbf{z})}^{\geq 2} \bmod \langle \mathbf{x} \rangle^{t+1} \subseteq \mathbb{F}(\mathbf{z})[\mathbf{x}].$$

Alternate Statement for the PSS Criterion

$\{f_1, f_2, \ldots, f_k\}$ is algebraically independent if and only if for every $(v_1, v_2, \ldots, v_k)$ with v_is in $\mathcal{I}_t$,

$$\mathcal{H}(\mathbf{f}, \mathbf{v}) = \begin{bmatrix} \ldots & \mathcal{H}_t(f_1) + v_1 & \ldots \\ \ldots & \mathcal{H}_t(f_2) + v_2 & \ldots \\ & \vdots & \\ \ldots & \mathcal{H}_t(f_k) + v_k & \ldots \end{bmatrix}$$ has full rank over $\mathbb{F}(\mathbf{z})$.

The Goal

Define: $\mathcal{F} = \{\varphi : \{x_1, \ldots, x_n\} \to \mathbb{F}[y_1, \ldots y_k]\}$ with $|\mathcal{F}| \approx \text{poly}(n)$.

Required: $\mathcal{C} = \mathcal{C}'(f_1, \ldots, f_m) \neq 0$, $\text{algrank}(f_1, \ldots, f_m) \leq k$

$\implies$ for some $\varphi \in \mathcal{F}$, $\mathcal{C}_0 = \mathcal{C}'(f_1 \circ \varphi, \ldots, f_m \circ \varphi) \neq 0$.

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Sufficient: Let $g_i = f_i \circ \varphi$ and $\mathcal{H}_t(g) = \deg^{\leq t}(g(\mathbf{y} + \mathbf{v}) - g(\mathbf{v}))$.
Then,

$$\mathcal{H}(\mathbf{f} \circ \varphi, \mathbf{u}) = \begin{bmatrix} \ldots & \mathcal{H}_t(f_1 \circ \varphi) + u_1 & \ldots \\ \ldots & \mathcal{H}_t(f_2 \circ \varphi) + u_2 & \ldots \\ & \vdots & \\ \ldots & \mathcal{H}_t(f_m \circ \varphi) + u_m & \ldots \end{bmatrix}$$

has full rank for every $u_1, u_2, \ldots, u_m \in \mathcal{I}_t(\varphi)$ where

$$\mathcal{I}_t(\varphi) = \langle \mathcal{H}_t(g_1), \ldots, \mathcal{H}_t(g_m) \rangle^{\geq 2}_{\mathbb{F}(\mathbf{g}(\mathbf{v}))} \bmod \langle \mathbf{y} \rangle^{t+1} \subseteq \mathbb{F}(\mathbf{v})[\mathbf{y}].$$

Constructing Faithful Homomorphisms over Arbitrary Fields

What to ask for from the map

$$\varphi : x_i \to \sum_{j=1}^{k} a_{ij} y_j + b_i y_0 \qquad \text{and} \qquad \varphi_z : z_i \to \sum_{j=1}^{k} a_{ij} w_j + b_i w_0$$

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Sufficient Properties

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4. M_φ preserves rank

The Matrix Decomposition

$$\left[\quad \mathcal{H}(\mathbf{f}\circ\varphi,\mathbf{u}) \quad\right] = \overbrace{\left[\quad \varphi_z(\mathcal{H}(\mathbf{f},\mathbf{v})) \quad\right]}^{\text{labelled by } \mathbf{x^e}} \times \underbrace{\left[\quad M_\varphi \quad\right]}_{\text{labelled by } \mathbf{y^d}}$$

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where

$$M_\varphi(\mathbf{x^e},\mathbf{y^d}) = \begin{cases} \text{coeff}_{\mathbf{y^d}}(\varphi(\mathbf{x^e})) & \text{if } \sum e_i = \sum d_i \\ 0 & \text{otherwise} \end{cases}$$

What makes Vandermonde type matrices work?

$$\begin{bmatrix} & A & \end{bmatrix} \times \begin{bmatrix} & M & \end{bmatrix} = \begin{bmatrix} & AM & \end{bmatrix}$$

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$$
\begin{array}{c}
x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x_n
\end{array}
\begin{bmatrix}
\left(s^{\mathrm{wt}(x_1)}\right)^1 & \cdots & \left(s^{\mathrm{wt}(x_1)}\right)^k \\
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\vdots & & \vdots \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
\vdots & & \vdots \\
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\end{bmatrix}
$$

$\mathrm{wt}(x_i)$ is distinct for each i

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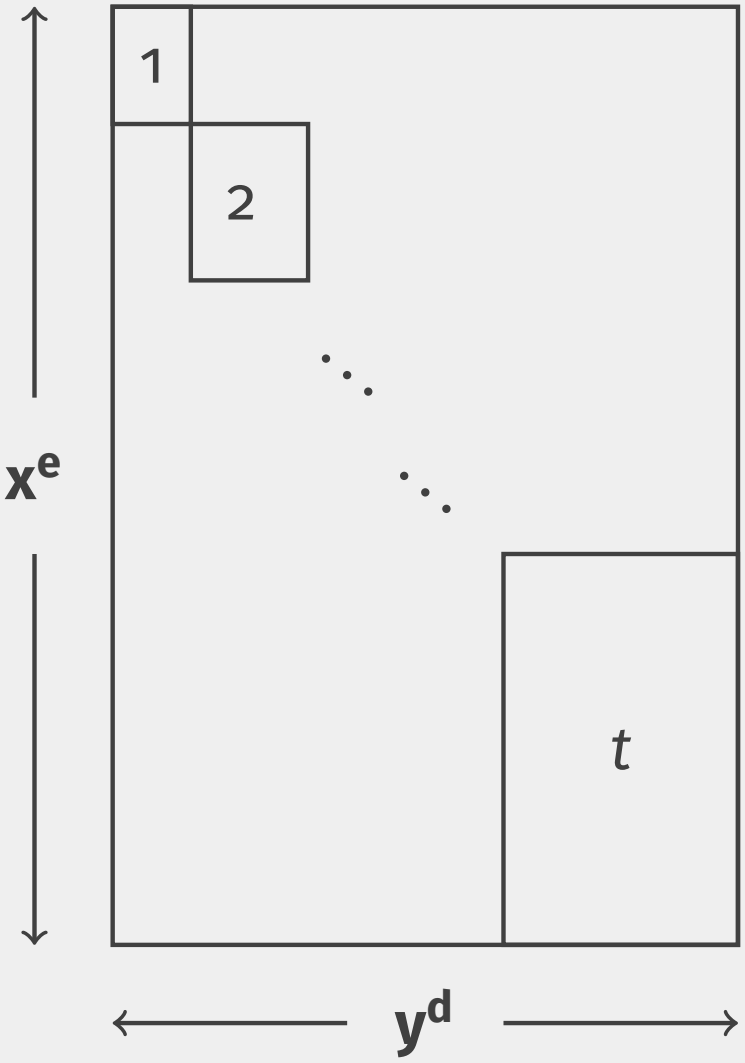

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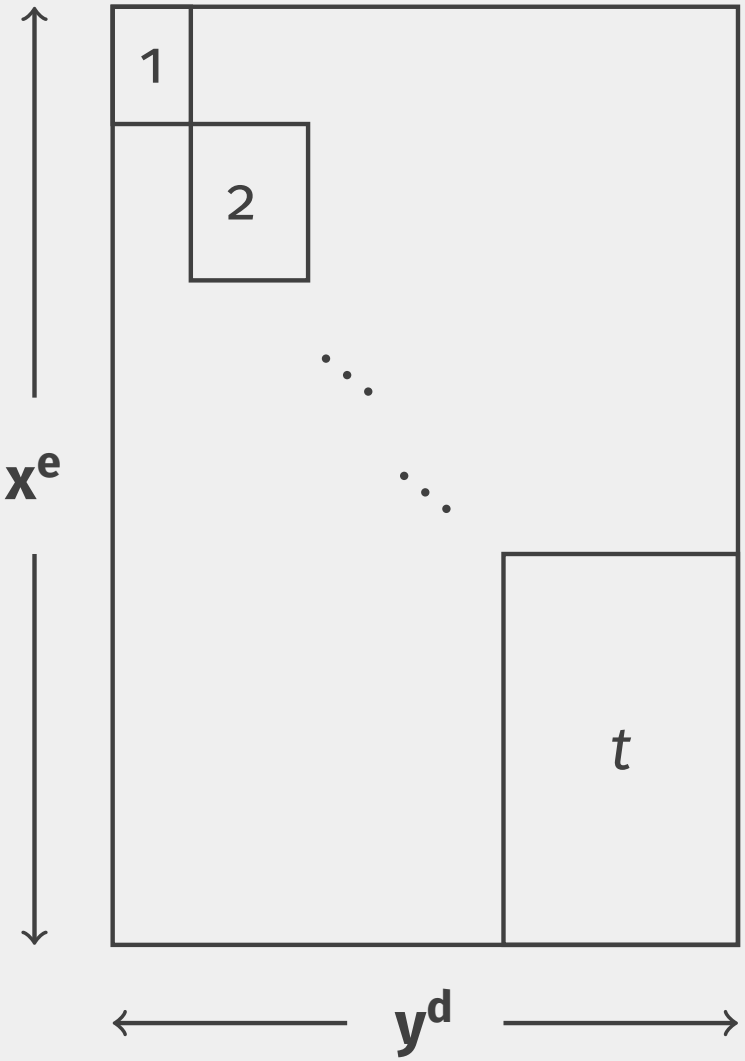

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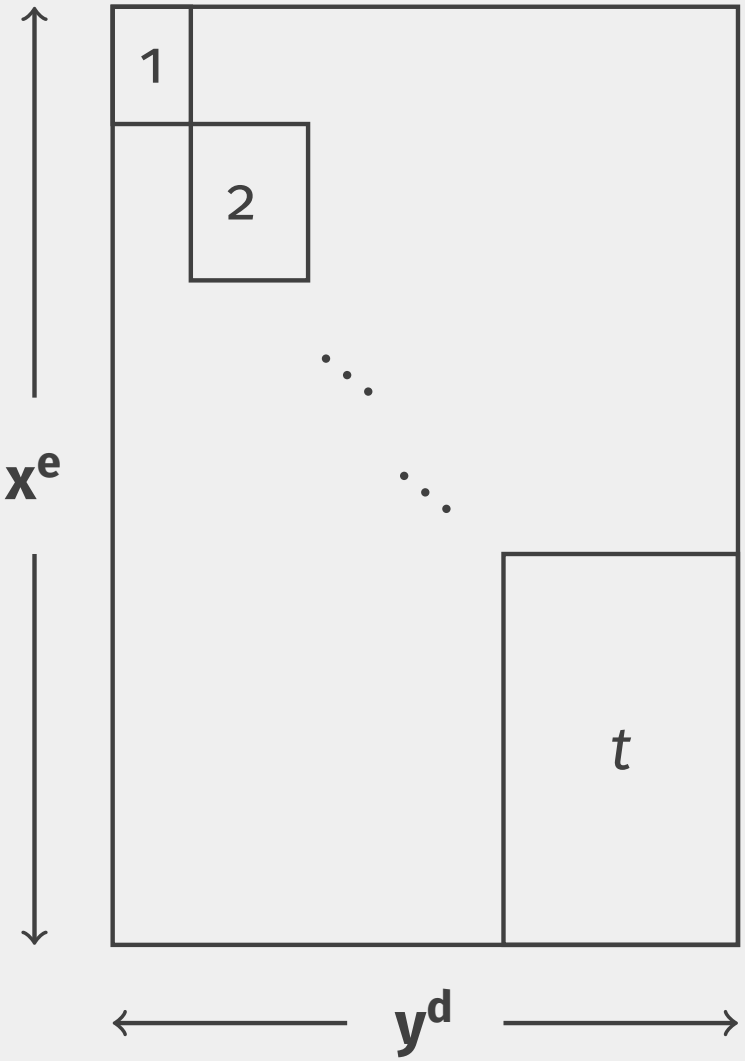

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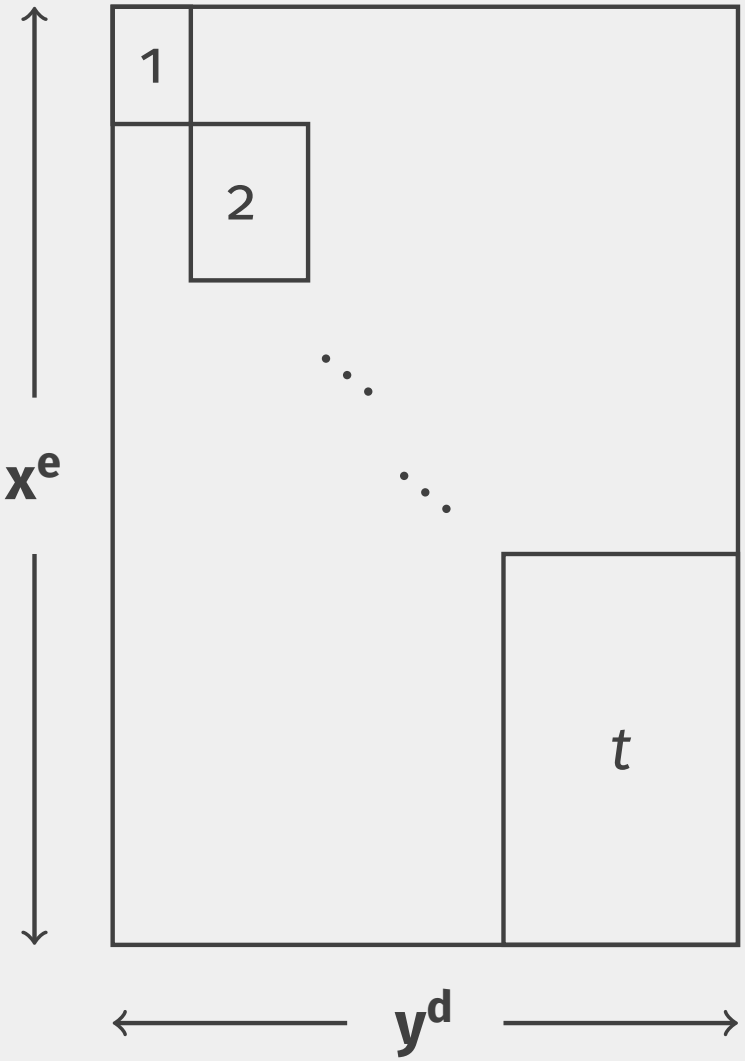

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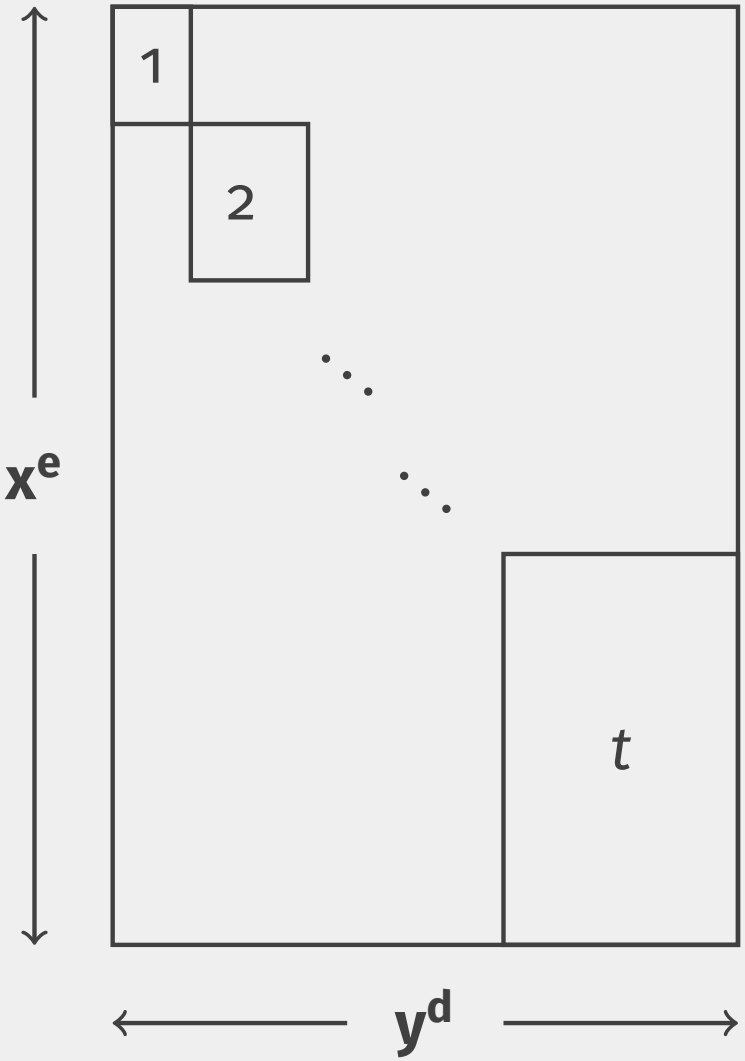

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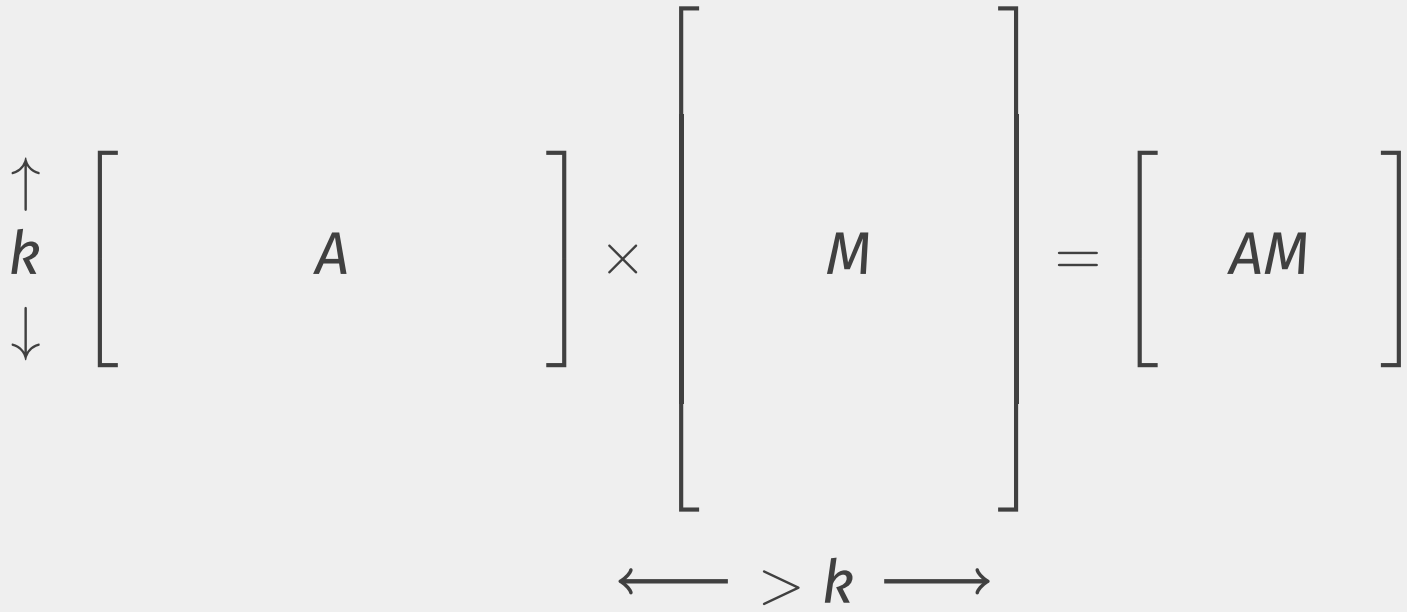
$$\text{If } B = (\mathbf{x}^{\mathbf{e}_1}, \mathbf{x}^{\mathbf{e}_2}, \ldots, \mathbf{x}^{\mathbf{e}_k}),$$

$$\text{then } \text{wt}(B) = \sum_{j\in[k]} j \,\text{wt}(\mathbf{x}^{\mathbf{e}_j})$$

A Rank Preserving Matrix

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix} \times \begin{bmatrix} \\ \\ M \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ AM \\ \\ \end{bmatrix}$$

A Rank Preserving Matrix



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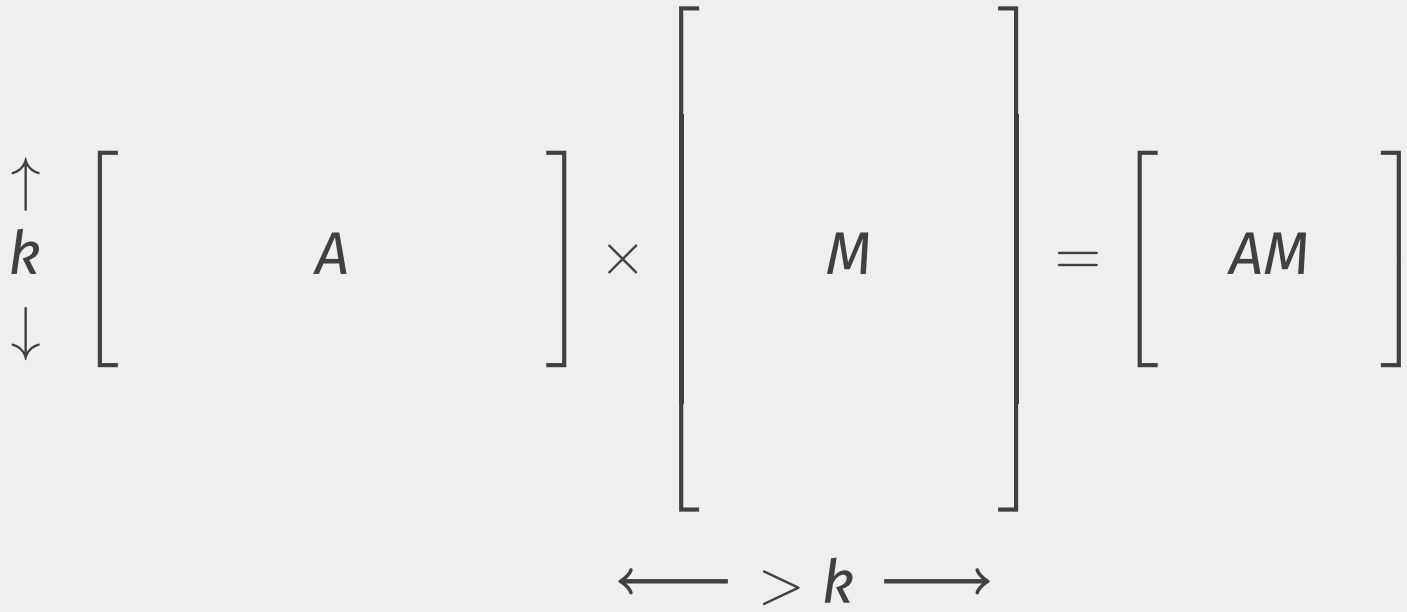
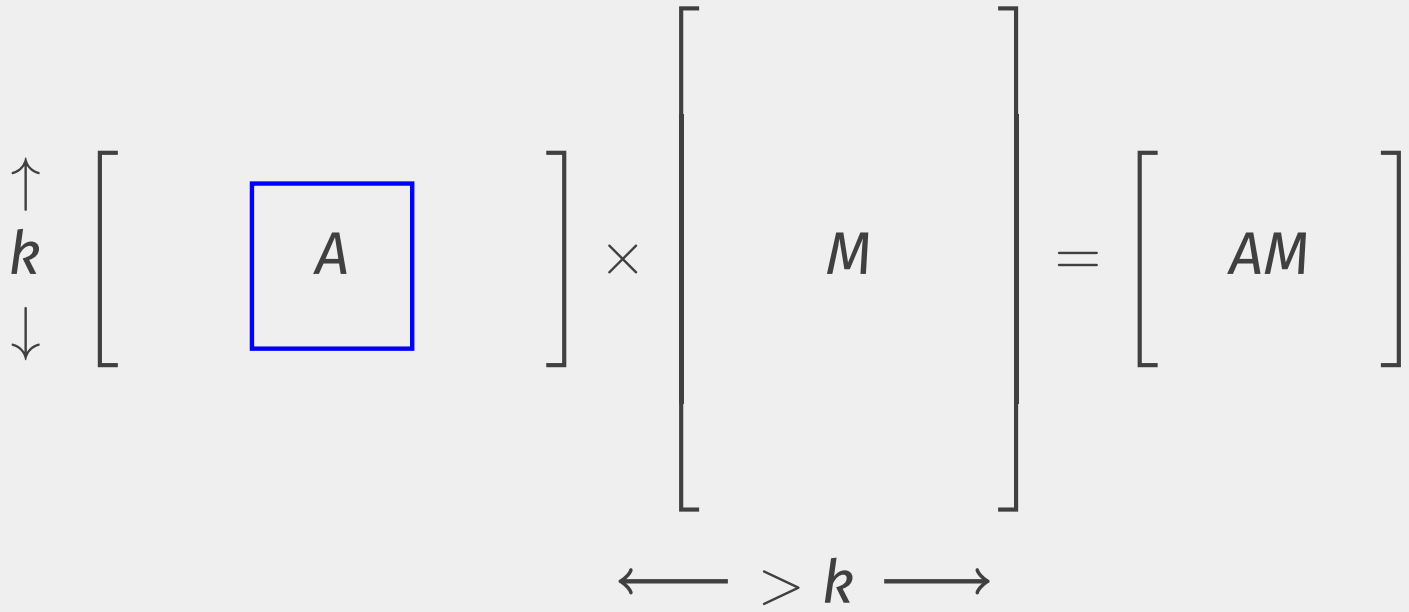


What we want: k columns of AM that are linearly independent.

A Rank Preserving Matrix



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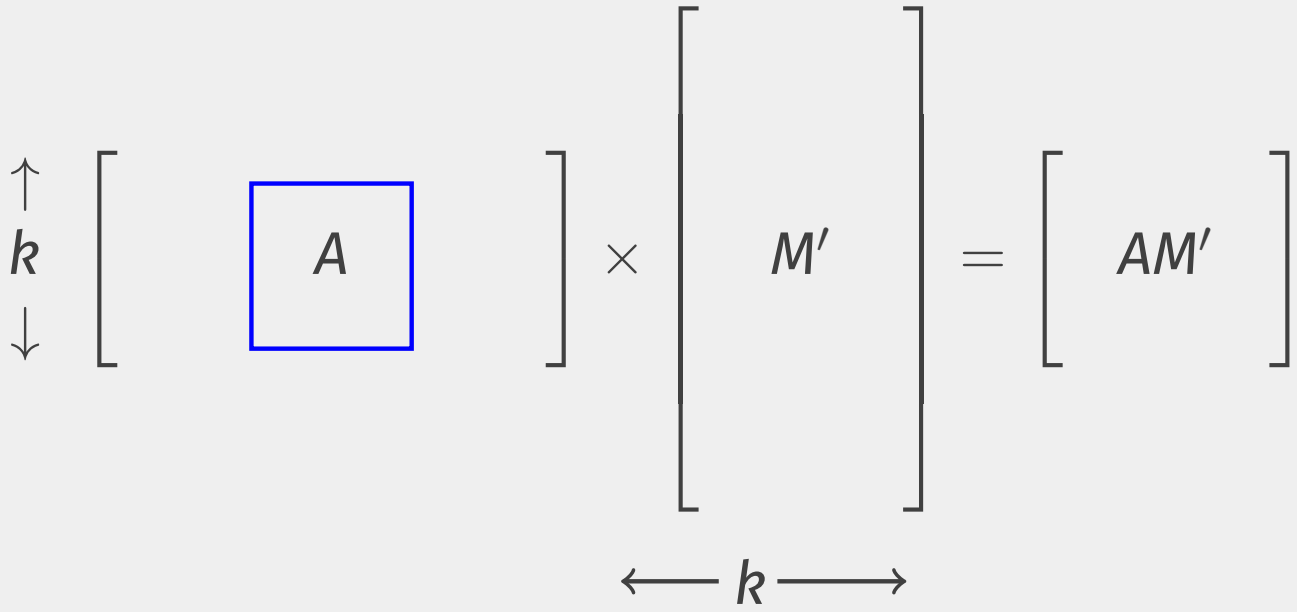

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Proof Strategy:

- Isolate a unique non-zero minor A_{B_0} with maximum weight
- $M' \equiv k$ columns of M such that $\deg_s(\det(M'_{B_0})) = \mathrm{wt}(B_0)$

A few details

About $\deg_s(\det(M'_{B_0}))$ **for** $B \neq B_0$:

- $\deg_s(\det(M'_B)) \leq \mathrm{wt}(B)$ for $B \neq B_0$

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About M'

- M' can always be chosen such that its columns are indexed by "pure" monomials.

A Faithful Map over Arbitrary Fields

$$\varphi : x_i \to \sum_{j=1}^{k} s^{\mathrm{wt}(i)j} y_j + a_i y_0 \quad \text{and} \quad \varphi_z : z_i \to \sum_{j=1}^{k} s^{\mathrm{wt}(i)j} w_j + a_i y_0$$

where t is the inseparable degree and $\mathrm{wt}(i) = (t+1)^i \bmod p$.

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Open Threads

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Thank you!